Covariant Lagrangian Formulation of Chern-Simons and BF Theories

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Abstract: We investigate the covariant formulation of Chern-Simons theories in a general odd dimension which can be obtained by introducing a vacuum connection field as a reference. Field equations, Nöther currents and superpotentials are computed so that results are easily compared with the well-known results in dimension 3. Finally we use this covariant formulation of Chern-Simons theories to investigate their relation with topological BF theories.

1. Introduction

Chern-Simons theories are, in their original formulation, global field theories defined by sheafs of local Lagrangians depending on some (principal) connection. The local Lagrangians on two intersecting patches differ in the patch intersection by a local divergence which depends on the gauge choice. Despite these local divergences prevent the local Lagrangians to glue together and form a global Lagrangian, the local Lagrangians altogether define in each patch the same gauge-covariant field equations which are thence the appropriate restrictions of a single globally well-defined field equation.

However, a globalization procedure is usually defined only after another connection has been fixed. This allows one to define a global Lagrangian depending on two connections (together with their curvatures) which is gauge covariant and induces the same (global) field equations of the original local Lagrangian sheaf.

Conservation laws are particularly tricky when local Lagrangians are considered. Due to the local divergence indetermination, the local Lagrangians are not gauge invariant. They are just locally invariant modulo local divergences. In this situation the application of Nöther theorem is tricky and the definition of Nöther currents needs some extra care. On the contrary, when a global covariant Lagrangian is used these problems disappear

altogether and superpotentials can be computed as the theory is now set as a global gauge natural theory (see [**1**], [**2**], [**3**], [**4**]).

The situation was preliminary discussed in detail in dimension 3 (see [**5**], [**6**], [**7**] and references quoted therein). It will be discussed here in any arbitrary odd dimension. As it should be expected, the explicit expression for a covariant global Lagrangian grows in complication with dimension; these expressions are however interesting since they present a family of global topological theories. In this paper we shall explicitly consider the case of Chern-Simons Lagrangians for gauge theories; we shall not deal with Chern-Simons gravitational models, which will be treated elsewhere (see [**8**]).

2. Notation

Let $\mathcal{P} = (P, M, \pi, G)$ be a principal bundle. Even if the notation $\mathcal{P}$ and P should be in principle reserved to denote respectively the bundle and its total space, we shall, by an abuse of language (and for simplicity), consider them as being equivalent. Let $\mathfrak{g}$ denote the Lie algebra of the structure group G and T_A be a basis of $\mathfrak{g}$. If the group is semisimple then T_A is usually chosen to be orthonormal with respect to the Cartan-Killing metric of G.

We shall hereafter denote by $\mathrm{ad} : G \times G \to G : (g, h) \mapsto g \cdot h \cdot g^{-1}$ the adjoint action of the group onto itself, by $T_e\mathrm{ad} \equiv \mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g} : (g, T_A) \to \mathrm{Ad}^B_A(g)T_B$ the induced adjoint action of the group onto the Lie algebra, and by $\mathbf{Ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : [T_A, T_B] \mapsto c_{AB}{}^C T_C$ the induced adjoint action of the Lie algebra onto itself. The constants $c_{AB}{}^C$ are the *structure constants* of the group G; they depend of course on the basis T_A.

Fibered coordinates on $\mathcal{P}$ are denoted by (x^μ, g^a); the Lie algebra is identified with the tangent space T_eG to the group at the identity $e \in G$. Accordingly, we set $T_A = T^a_A \partial_a|_e$ (where $\partial_a|_e$ denotes the basis of tangent vectors to G induced by the coordinates chosen around the identity e). We denote by $R_g : P \to P : p \mapsto p \cdot g$ the canonical right action of the group G on the principal bundle P and by $T_eR_g : \mathfrak{g} \to \mathfrak{g}$ the induced right action on the algebra given by $TR_g(T_A) =: R^B_A(g)T_B =: R^a_A(g)\partial_a|_e$. Let ρ_A denote a right invariant pointwise basis of vertical vectors corresponding to the basis T_A. One has $\rho_A = R^a_A(g)\partial_a$.

As it is standard in the gauge natural framework (see [**1**]) $W^{(1,1)}G$ will denote the semidirect product $W^{(1,1)}G = \mathrm{GL}(m) \rtimes J^1G$ and the fibered product $W^{(1,1)}\mathcal{P} = L(M) \times_M J^1\mathcal{P}$ denotes the gauge natural prolongation of order $(1, 1)$ of the principal bundle $\mathcal{P}$ (see [**1**],

[**9**]). The bundle $W^{(1,1)}\mathcal{P}$ is a principal bundle with $W^{(1,1)}G$ as structure group. If (x^μ, V^μ_a) are fibered coordinates of $L(M)$, then $(x^\mu, V^\mu_a, g^a, g^a_\mu)$ are fibered coordinates on $W^{(1,1)}\mathcal{P}$.

Let us now set $\mathbb{A} = (\mathbb{R}^m)^* \otimes \mathfrak{g}$ (considered as an affine space with local coordinates A^A_μ) and let us consider the following affine action on the left:

$$\lambda : W^{(1,1)}G \times \mathbb{A} \to \mathbb{A} : (J, g, dg, A) \mapsto (\mathrm{Ad}^A_B(g) A^B_\nu + \bar{R}^A_b(g) g^b_\nu)\bar{J}^\nu_\mu \tag{2.1}$$

where the bars denote matrix inversion. The associated bundle $\mathcal{C}(P) := W^{(1,1)}\mathcal{P} \times_\lambda \mathbb{A}$ is known to be canonically isomorphic to $J^1\mathcal{P}/G$. Its global sections are in one-to-one canonical correspondence with principal connections on $\mathcal{P}$, which we generically denote by A. We stress that, by definition, $\mathcal{C}(P)$ is a bundle over M, despite principal connections in mathematical literature usually *live* on P and their corresponding objects on M are obtained by gauge fixing. Our representation for principal connections is instead done in terms of sections of a bundle over M. Nevertheless, it is manifestly global and gauge covariant and it does not rely on any gauge fixing.

Given a principal connection on $\mathcal{P}$ under its (local) form $\omega = \mathrm{d}x^\mu \otimes (\partial_\mu - A^A_\mu(x)\,\rho_A)$, its *curvature* is given by

$$F = \tfrac{1}{2} F^A_{\mu\nu} T_A \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \qquad F^A_{\mu\nu} = \mathrm{d}_\mu A^A_\nu - \mathrm{d}_\nu A^A_\mu + c_{BC}{}^A A^B_\mu A^C_\nu \tag{2.2}$$

Let us now denote by $\Lambda^k(P;\mathfrak{g})$ the bundle of k-forms on M with values in the bundle $\Lambda^0(P;\mathfrak{g})$, for any positive integer k. For $k = 0$ we obtain $\Lambda^0(P;\mathfrak{g})$, i.e. the bundle of (infinitesimal) gauge transformations. The bundle $\Lambda^k(P;\mathfrak{g})$ is associated to $L(M) \times_M P$ through the following action on $V = \Lambda^k(\mathbb{R}^m) \otimes \mathfrak{g}$:

$$l : \mathrm{GL}(m) \times G \times V \mapsto V : (J, g, \omega) \mapsto \mathrm{Ad}^A_B(g)\, \omega^B_{\rho\sigma\ldots\theta} \bar{J}^\rho_\mu \bar{J}^\sigma_\nu \ldots \bar{J}^\theta_\lambda \tag{2.3}$$

The bundles $\Lambda^k(P;\mathfrak{g})$ are by construction gauge natural bundles of order $(1,0)$. $\Lambda^1(P;\mathfrak{g})$ is the vector bundle which the affine connection bundle $\mathcal{C}(P)$ is modeled on; the difference of two connections A and $\bar{A}$ is a section of $\Lambda^1(P;\mathfrak{g})$, which we denote by $\alpha = \alpha(A, \bar{A}) = A - \bar{A}$. The curvature of a connection is a section of $\Lambda^2(P;\mathfrak{g})$. Equation (2.2) is in fact the local expression of a global bundle morphism $\mathbf{F} : J^1\mathcal{C}(P) \to \Lambda^2(P;\mathfrak{g})$.

Let us remark that, by construction, our bundles $\Lambda^k(P;\mathfrak{g}) \cong \Lambda^k(M) \otimes \Lambda^0(P;\mathfrak{g})$ are bundles over M, not to be confused with the bundles $\Lambda^k(P) \otimes \mathfrak{g}$ of $\mathfrak{g}$-valued k-forms on P (here $\mathfrak{g}$ can be identified with the trivial bundle $P \times \mathfrak{g}$) used frequently in literature (e.g. [**10**]) and often denoted in a similar way (e.g. [**9**]) which, on the contrary, are defined as bundles

over P. In particular, $\Lambda^0(P) \otimes \mathfrak{g} \cong P \times \mathfrak{g}$, which coincides with the vertical bundle of the principal bundle P, is always a trivial vector bundle over P, while our $\Lambda^0(P;\mathfrak{g})$ is, in general, a nontrivial bundle over M. More generally, there is a one-to-one correspondence between usual sections of $\Lambda^k(P;\mathfrak{g})$ and the so-called *tensorial* $\mathfrak{g}-$valued $k-$forms on P, i.e. *tensorial* sections of $\Lambda^k(P) \otimes \mathfrak{g}$ **[10]**. We stress that any standard connection 1-form is a vertical and Ad$-$invariant section of the bundle $\Lambda^1(P) \otimes \mathfrak{g}$ over P but it is not a global section of the bundle $\Lambda^1(P;\mathfrak{g})$ over M. To obtain the gauge fields often used in Physics (see **[10]**) which are *living* on M one needs to pull-back the connection 1-form along a section of P which exists globally if and only if P is trivial. This procedure is usually called a *gauge fixing* and we shall try to avoid it in order to work with objects which are already global in the most general situation.

The wedge product needs to be extended to forms with values in the Lie algebra (owing to the fact that the product of components needs to be replaced by Lie product, which is the only product in a generic Lie algebra). We define, e.g. on 1-forms

$$[\theta, \lambda] := c^A{}_{BC}\, \theta^B_\mu \lambda^C_\nu \, T_A \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \tag{2.4}$$

and similar more complicated expression hold for k-forms with $k > 1$. Notice in (2.4) the absence of combinatorial coefficients in the resulting 2-form. Let us also set for a k-form θ:

$$(\theta^2) := \tfrac{1}{(2k)!}[\theta, \theta] \tag{2.5}$$

which we stress to be non-zero for odd k. Hereafter we shall use this notation only in the case $k = 1$; e.g. when using (α^2).

The *exterior differential* of a k-form $\theta = \frac{1}{k!}\theta^A_{\mu_1\mu_2\ldots\mu_k} T_A \otimes \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \ldots \wedge \mathrm{d}x^{\mu_k}$ in $\Lambda^k(P;\mathfrak{g})$ is defined as

$$\mathrm{d}\theta = \tfrac{1}{k!}\mathrm{d}_\mu \theta^A_{\mu_1\mu_2\ldots\mu_k} \, T_A \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \ldots \wedge \mathrm{d}x^{\mu_k} \tag{2.6}$$

It is not gauge covariant; hence, once a connection A is fixed on P, the *exterior covariant differential* can be defined as

$$\nabla\theta = \mathrm{d}\theta + [A, \theta] \tag{2.7}$$

Of course a connection A is not a section of the bundle $\Lambda^1(P;\mathfrak{g})$ over M as eq. (2.7) should suggest (or, equivalently, connections are not *tensorial*; see **[10]**). Here a standard abuse of language is used to suitably identify a connection A with the (local) 1-form $\alpha = A - \bar{A}$ once, in a trivialization, a *trivial* background connection $\bar{A} = 0$ is set. Hereafter and

systematically in all local expressions we shall locally represent a connection by this section $\alpha_0 = \alpha(A, \bar{0}) = A - \bar{0} = A$ of $\Lambda^1(P; \mathfrak{g})$; accordingly we have the well-known expression

$$F = \mathrm{d}A + \frac{1}{2}[A, A] \tag{2.8}$$

for the curvature. By this abuse of language we shall earn a compact way for writing local expressions, even though the formulae so obtained are not completely meaningful from a global point of view. For example, despite the curvature (2.8) is globally well defined the terms $\mathrm{d}A$ and $\frac{1}{2}[A, A]$ are not separately global.

Let now $\Phi : P \to P$ be a principal automorphism projecting over a diffeomorphism $\phi : M \to M$. Its local expression is of the form

$$\begin{cases} x' = \phi(x) \\ g' = \varphi(x) \cdot g \end{cases} \tag{2.9}$$

where $\varphi(x)$ is a group isomorphism in G. The infinitesimal generator of a 1-parameter subgroup of principal automorphisms is a projectable right invariant vector field, i.e.:

$$\Xi = \xi^\mu(x)\partial_\mu + \xi^A(x)\rho_A \tag{2.10}$$

Principal automorphisms are also known as *gauge transformations*. They canonically induce automorphisms on any gauge natural bundle associated to $\mathcal{P}$, in particular on $\mathcal{C}(P)$ and all bundles $\Lambda^k(P; \mathfrak{g})$; the canonically induced vector fields on $\mathcal{C}(P)$ and $\Lambda^2(P; \mathfrak{g})$ are:

$$\mathcal{C}(\Xi) = \xi^\mu(x)\partial_\mu + \xi^A_\mu \partial^\mu_A, \qquad \xi^A_\mu = \mathrm{d}_\mu \xi^A + c_{BE}{}^A A^B_\mu \xi^E - \mathrm{d}_\mu \xi^\rho A^A_\rho \tag{2.11}$$

and:

$$\Lambda^2(\Xi; \mathfrak{g}) = \xi^\mu(x)\partial_\mu + \xi^A_{\mu\nu} \partial^{\mu\nu}_A, \qquad \xi^A_{\mu\nu} = c_{BC}{}^A \xi^B F^C_{\mu\nu} - \mathrm{d}_\mu \xi^\rho F^A_{\rho\nu} - \mathrm{d}_\nu \xi^\rho F^A_{\mu\rho} \tag{2.12}$$

respectively.

Hence we can define the Lie derivative of gauge natural objects with respect to an infinitesimal gauge generator Ξ. We easily obtain the following specific rules:

$$\pounds_\Xi A^A_\mu = \xi^\lambda F^A_{\lambda\mu} + \nabla_\mu(\xi^A_{(V)}) \tag{2.13}$$

and

$$\pounds_\Xi F^A_{\mu\nu} = \xi^\rho \nabla_\rho(F^A_{\mu\nu}) + \nabla_\mu \xi^\rho F^A_{\rho\nu} + \nabla_\nu \xi^\rho F^A_{\mu\rho} - c_{BC}{}^A \xi^B_V F^C_{\mu\nu} \tag{2.14}$$

where ∇_μ denotes covariant derivative with respect to the connection A^A_μ (as well as to a symmetric base connection $\Gamma^\alpha_{\beta\mu}$) and $\xi^A_{(V)} = \xi^A + A^A_\mu \xi^\mu$ denotes the vertical part of Ξ. Let us remark that the contribution of the symmetric base connection will be systematically cancelled out hereafter by symmetry reasons (e.g. in the exterior covariant differential below). More precisely we have:

$$\nabla_\mu(\xi^A_{(V)}) = \mathrm{d}_\mu \xi^A_{(V)} + c_{BC}{}^A A^B_\mu \xi^C_{(V)} \tag{2.15}$$

and

$$\nabla_\rho(F^A_{\mu\nu}) = \mathrm{d}_\rho F^A_{\mu\nu} - \Gamma^\lambda_{\mu\rho} F^A_{\lambda\nu} - \Gamma^\lambda_{\nu\rho} F^A_{\mu\lambda} + c_{BC}{}^A A^B_\rho F^C_{\mu\nu} \tag{2.16}$$

Using (2.2) and (2.16) one easily obtains the so-called *Bianchi identities*:

$$\nabla_{[\rho} F^A_{\mu\nu]} = \mathrm{d}_{[\rho} F^A_{\mu\nu]} + c_{BC}{}^A A^B_{[\rho} F^C_{\mu\nu]} = (c_{[DE}{}^C c_{B]C}{}^A) A^B_\rho A^D_\mu A^E_\nu \equiv 0 \tag{2.17}$$

where Jacobi identities in the Lie algebra have been implicitly used. An intrixic expression for Bianchi identies is obtained by means of the so-called *exterior covariant differential* with respect to the connection A^A_μ (see **[1]**, **[9]**, **[10]**):

$$\nabla F := \tfrac{1}{2} \nabla_{[\rho} F^A_{\mu\nu]}\, T_A \otimes \mathrm{d}x^\rho \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \equiv 0 \tag{2.18}$$

3. Invariant Polynomials

A *symmetric polynomial* on a vector space V is a symmetric k-linear form, k being any positive integer. We denote by $S^k(V) = \{f : V^k \to \mathbb{R} : f \text{ symmetric and } k\text{-linear}\}$.

If V happens to be the Lie algebra $V \equiv \mathfrak{g}$ of a Lie group G, then an *invariant polynomial* is a symmetric polynomial which is also Ad-invariant, i.e. $\forall g \in G$ and $\forall \xi_1, \xi_2, \ldots, \xi_k \in \mathfrak{g}$:

$$f(\mathrm{Ad}_g(\xi_1), \mathrm{Ad}_g(\xi_2), \ldots, \mathrm{Ad}_g(\xi_k)) = f(\xi_1, \xi_2, \ldots, \xi_k) \tag{3.1}$$

The set of all invariant polynomials of degree k is denoted by $I^k(\mathfrak{g})$. We set $\deg(f) := k$ for each $f \in I^k(\mathfrak{g})$.

The prototypes of invariant polynomials are the power-trace polynomials. By using the adjoint action $\mathbf{Ad}$ an element ξ of $\mathfrak{g}$ can be identified with an endomorphism $\mathbf{Ad}(\xi) : \mathfrak{g} \to \mathfrak{g} : \zeta \mapsto [\xi, \zeta] = \xi^A \zeta^B c_{AB}{}^C T_C$. The matrix representing such an endomorphism is:

$$\mathbf{Ad}^C_B(\xi) = \xi^A c_{AB}{}^C \tag{3.2}$$

We can in particular consider $f_k = \mathrm{Tr}(\mathbf{Ad}^k)$; for example:

$$f_2(\xi_1, \xi_2) = \xi_1^A \xi_2^B c_{AE}{}^D c_{BD}{}^E \in I^2(\mathfrak{g})$$
$$f_3(\xi_1, \xi_2, \xi_3) = \xi_1^A \xi_2^B \xi_3^C c_{AE}{}^D c_{BF}{}^E c_{CD}{}^F \in I^3(\mathfrak{g})$$

and so on.

These polynomials are trivially symmetric and Ad-invariant; one can easily check that in specific examples they are not trivially vanishing (e.g. when $G = \mathrm{SO}(3)$). Before entering in details let us first briefly summarize the main points of the construction leading to Chern-Simons theories (see **[11]**, **[12]**).

Once a connection A has been fixed on P there exists a canonical prescription that to any invariant polynomial $f \in I^k(\mathfrak{g})$ associates a closed horizontal $2k$-form on $\mathcal{P}$ denoted by $f(F^k) \in \Lambda^{2k}(P)$. This forms is *transgressive*, i.e. it is the pull-back along the projection $\pi : P \to M$ of a $2k$-form $\mathbf{f}(F^k)$ on M (see **[13]**). Despite the form $\mathbf{f}(F^k)$ depends on the connection A one can check that the cohomology class it identifies in $H^{2k}(M, \mathbb{R})$ is independent of A. In fact, if any two connections A and $\bar{A}$ are fixed on P one is then able to *constructively* single out a representative $T\mathbf{f}(A, \bar{A})$ of this cohomology class. This is a global $(2k-1)$-form on M. Hence a local potential $T\mathbf{f}(A) = T\mathbf{f}(A, 0)$ (which is global on trivial bundles) is singled out on M for the form $\mathbf{f}(F^k)$.

If the dimension of M is an odd integer $m = 2k-1$, $k := \deg(f)$, then $T\mathbf{f}(A, \bar{A})$ is a global closed $(2k-1)$-form and the local forms $T\mathbf{f}(A)$ can be used as a sheaf of local Lagrangians; we shall prove that these *local* Lagrangians induce *global* field equations.

We remark that $T\mathbf{f}(A, \bar{A})$ will be used as a convenient globalization of the local Lagrangians $T\mathbf{f}(A)$. As it often happens, globality is obtained by introducing a background (or, using a better terminology introduced in **[14]**, a *reference vacuum state*) and mimicking the covariant first order Lagrangian known to exist for General Relativity (see **[14]**, **[15]**, **[16]**).

Let us also stress that the bundle level P is essential to the construction. Once again (as, e.g., for superpotentials) objects have no canonical representative at the base level, though canonical representatives can be defined at the bundle level and eventually be pulled back on the base manifold. For example, in the case of interest $m = 2k-1$, the form $f(F^k)$ on M identically vanishes due to dimensional reasons. The potential $T\mathbf{f}(A, \bar{A})$ we shall build is not zero (as one could expect by working in the base manifold). It is important to regard $T\mathbf{f}(A, \bar{A})$ as a map defined in general for all k, all m and all topologies of P, to be later specialized to particular cases (even when for other reasons other *canonical representatives* exist).

Let us hence begin to define the correspondence between invariant polynomials and bundle forms in the general case ($\dim(M) = m$ and k any pair of positive integers). Let $f \in I^k(\mathfrak{g})$ be an invariant polynomial of degree k; let us denote by $f_{A_1 A_2 \dots A_k} := f(T_{A_1}, T_{A_2}, \dots, T_{A_k})$ its symmetric coefficients. Since the curvature $F \in \Lambda^2(P; \mathfrak{g})$ is a $\mathfrak{g}$-valued 2-form on P we set

$$f(F^k) = \frac{1}{2^k} f_{A_1 A_2 \dots A_k} F^{A_1}_{\mu_1 \nu_1} F^{A_2}_{\mu_2 \nu_2} \dots F^{A_k}_{\mu_k \nu_k} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\nu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \mathrm{d}x^{\nu_2} \wedge \dots \wedge \mathrm{d}x^{\mu_k} \wedge \mathrm{d}x^{\nu_k} \quad (3.3)$$

These local expressions glue together (because of the transformation laws (2.3) and the Ad-invariance of the polynomial f) to uniquely determine a global form on $\Lambda^{2k}(P)$. The same expression defines a $2k$-form on M since for a specific connection its curvature coefficients depend on spacetime coordinates x alone. We remark that it may happen (for large enough k) that such a form is identically zero.

From now on we shall denote by $f(\alpha_1, \dots, \alpha_p, \beta^{k-p}) = f(\alpha_1, \dots, \alpha_p, \beta, \beta \dots, \beta)$, $0 \leq p \leq k$, with the last argument β repeated $(k-p)$ times. Accordingly, we have $f(F^k) = f(F, F, \dots, F)$, with F repeated k times.

Property (3.4): the form $f(F^k)$ is closed.

Proof: *it follows from Bianchi identities $\nabla F = 0$:*

$$d(f(F^k)) = \nabla(f(F^k)) = k f(\nabla F, F^{k-1}) = 0 \quad (3.5)$$

■

Property (3.6): the cohomology class $[f(F^k)] \in H^{2k}(M, \mathbb{R})$ is independent of the connection A.

Proof: *let us consider two connections A and $\bar{A}$ on $\mathcal{P}$. The difference $\alpha = A - \bar{A}$ is a well-defined global section of $\Lambda^1(P; \mathfrak{g})$ due to the affine structure on the bundle of connections $\mathcal{C}(P)$, which is modelled precisely on $\Lambda^1(P; \mathfrak{g})$. Let $\omega_s = \bar{A} + s\alpha$, $s \in \mathbb{R}$ be the so-called* interpolating connection *(again notice that the affine structure on $\mathcal{C}(P)$ ensures that ω_s is in fact a 1-parameter family of global connections on $\mathcal{P}$). Let us denote by Ω_s the corresponding curvature, given by:*

$$\Omega_s = \bar{F} + s\bar{\nabla}\alpha + s^2 \, (\alpha^2) \quad (3.7)$$

where, following the notation of Appendix A, we set

$$\begin{aligned} \Omega_s &:= d\omega_s + \tfrac{1}{2}[\omega_s, \omega_s] \qquad & \bar{\nabla}\alpha &:= d\alpha + [\bar{A}, \alpha] \\ \bar{F} &:= d\bar{A} + \tfrac{1}{2}[\bar{A}, \bar{A}] \qquad & (\alpha)^2 &:= \tfrac{1}{2}\,[\alpha, \alpha] \end{aligned} \tag{3.8}$$

Moreover one easily obtains:

$$\frac{d}{ds}\Omega_s = \bar{\nabla}\alpha + s[\alpha, \alpha] =: \nabla^s \alpha \tag{3.9}$$

where $\nabla^s \alpha$ is the exterior covariant differential induced on $\Lambda^1(P; \mathfrak{g})$ by the connection ω_s.

Thus we have:

$$\begin{aligned} f(F^k) - f(\bar{F}^k) =& \int_0^1 \frac{d}{ds} f(\Omega_s^k) ds = k \int_0^1 f\left(\frac{d}{ds}\Omega_s^k, \Omega_s^{k-1}\right) ds = \\ =& k \int_0^1 f(\nabla^s \alpha, \Omega_s^{k-1}) ds = k \int_0^1 \nabla^s f(\alpha, \Omega_s^{k-1}) ds = \\ =& k \int_0^1 d f(\alpha, \Omega_s^{k-1}) ds = d\left(k \int_0^1 f(\alpha, \Omega_s^{k-1}) ds\right) \end{aligned} \tag{3.10}$$

It then follows $[f(F^k)] = [f(\bar{F}^k)]$. ■

Let us now set $T\mathbf{f}(A, \bar{A}) := k \int_0^1 f(\alpha, \Omega_s^{k-1}) \mathrm{d}s$. For two specific connections A and $\bar{A}$ this is a $(2k-1)$-form on P, though the same expression defines a $(2k-1)$-form on M as well (which will be denoted below by $\mathbf{Tf}(A, \bar{A})$). Given a background reference connection $\bar{A}$ we are then able to single out a canonical representative $f(\bar{F}^k) + \mathrm{d}T\mathbf{f}(A, \bar{A})$ for the cohomology class $[f(F^k)]$.

Locally on P, once a trivialization is chosen, we can set $\bar{A} = 0$ (which of course has no global meaning) and obtain

$$f(F^k) = \mathrm{d}\, T\mathbf{f}(A, 0) \equiv \mathrm{d}\, T\mathbf{f}(A) \tag{3.11}$$

which is a local potential (on M) for the form $f(F^k)$.

Notice that if $m = 2k-1$ is odd both $f(F^k)$ and $f(\bar{F}^k)$ are zero and $T\mathbf{f}(A, \bar{A})$ is a global m-form over P; it is also transgressive, i.e. $T\mathbf{f}(A, \bar{A}) = \pi^* \mathbf{Tf}(A, \bar{A})$ for some m-form over M, $\mathbf{Tf}(A, \bar{A})$ having the same local expression of $T\mathbf{f}(A, \bar{A})$. We shall see in the next Section that both Lagrangians $\mathbf{Tf}(A, \bar{A})$ and $\mathbf{Tf}(A) = \mathbf{Tf}(A, 0)$ produce the same global

field equations (even if the second is just a family of local Lagrangians). As a consequence $\mathbf{Tf}(A, \bar{A})$ is by construction the globalization of $\mathbf{Tf}(A)$ obtained by introducing a reference vacuum state.

Property (3.12): the form $f(F^k)$ is transgressive.

Proof: *it follows immediately from the local expression* (3.3). *The form $f(F^k)$ is fiberwise constant, hence projectable onto a form $\mathbf{f}(F^k)$ on M given by the same local expression as* (3.3) *(we stress that the connection has been fixed so that $F^A_{\mu\nu}(x)$ are fuctions of the variables x only). One trivially has $f(F^k) = \pi^* \mathbf{f}(F^k)$.* ∎

4. Variational Properties of Chern-Simons Lagrangians

From now on let us set $m = 2k-1$. The form $T\mathbf{f}(A, \bar{A})$ on P can be canonically regarded as the evaluation of a horizontal form on $J^1\mathcal{C}(P)$ along the 1-jet prolongation of a section A of the bundle $\mathcal{C}(P)$ of connections. By an abuse of notation we shall consider the global Lagrangian (see also **[13]** and references quoted therein)

$$L^{(k)}_{A\bar{A}} = \mathbf{Tf}(A, \bar{A}) = k \int_0^1 f(\alpha, \Omega_s^{k-1})\, \mathrm{d}s \tag{4.1}$$

and the local Lagrangians

$$L^{(k)}_{A} = \mathbf{Tf}(A) = k \int_0^1 f(A, (sF + \tfrac{1}{2}s(s-1)[A, A])^{k-1})\, \mathrm{d}s \tag{4.2}$$

where $[A, A] = c_{BC}{}^{A} A^B_\mu A^C_\nu \, T_A \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ (which of course makes sense only in a trivialization). If the trivialization is changed the connection transforms according to its affine rules (2.1) so that the polynomial $\mathbf{Tf}(A)$ (and consequently the Lagrangian $L^{(k)}_{A}$) is not global.

The first variation formula for the Lagrangian $L^{(k)}_{A\bar{A}}$ can be obtained:

$$\delta L^{(k)}_{A\bar{A}} = \mathbb{E}(L^{(k)}_{A\bar{A}}) + \mathrm{d}\mathbb{F}(L^{(k)}_{A\bar{A}}) \tag{4.3}$$

where we set

$$\begin{cases} \mathbb{E}(L^{(k)}_{A\bar{A}}) = k f(\delta A, F^{k-1}) - k f(\delta \bar{A}, \bar{F}^{k-1}) \\ \mathbb{F}(L^{(k)}_{A\bar{A}}) = k(k-1) \int_0^1 f(\delta\omega_s, \alpha, \Omega_s^{k-2})\, \mathrm{d}s \end{cases} \tag{4.4}$$

A detailed derivation of these results can be found below in Appendix B.

From these results one can obtain as a local special case the first variation formula for the local Lagrangians $\overset{(k)}{L}_A$

$$\delta \overset{(k)}{L}_A = \mathbb{E}(\overset{(k)}{L}_A) + \mathrm{d}\mathbb{F}(\overset{(k)}{L}_A) \tag{4.5}$$

by simply inserting $\bar{A} = 0$ into the equations (4.4), i.e.

$$\begin{cases} \mathbb{E}(\overset{(k)}{L}_A) = k f(\delta A, F^{k-1}) \\ \mathbb{F}(\overset{(k)}{L}_A) = k(k-1)\int_0^1 f(s\delta A, A, (sF + s(s-1)(A^2))_s^{k-2})\,\mathrm{d}s = \\ \qquad\qquad = \sum_{i=0}^{k-2} \frac{(-1)^i k!(k-1)!}{(k-i-2)!(k+i)!} f(\delta A, A, (A^2)^i, F^{k-i-2}) \end{cases} \tag{4.6}$$

We stress (referring again to the Appendix B) that specializing $k = 2, 3, \ldots$ one obtains the standard results obtained in any odd dimension $m = 3, 5, \ldots$ (see [**17**]).

We can now apply the results of [**18**] (in their form summarized in Appendix A) in order to split the covariant Lagrangian $\overset{(k)}{L}_{A\bar{A}}$ as

$$\overset{(k)}{L}_{A\bar{A}} = \overset{(k)}{L}_A - \overset{(k)}{L}_{\bar{A}} + \mathrm{d}\overset{(k)}{\Delta} \tag{4.7}$$

where $\overset{(k)}{\Delta}$ is the form

$$\overset{(k)}{\Delta} = -\sum_{i=0}^{k-2} \frac{(-1)^i k!(k-1)!}{(k-i-2)!(k+i)!} \int_0^1 f(\alpha, \omega_s, (\omega_s^2)^i, (\Omega_s)^{k-i-2})\,\mathrm{d}s \tag{4.8}$$

5. Conserved Quantities

Both α and Ω_s are sections of $\Lambda^k(P;\mathfrak{g})$ (for $k = 1$ and $k = 2$, respectively) and hence they transform according to (2.3) under principal automorphisms of the principal bundle P ([**5**], [**6**], [**19**], [**20**]). Being f Ad-invariant the Lagrangian $\overset{(k)}{L}_{A\bar{A}}$ is Aut(P)-covariant. As a consequence (as one can also check directly by expanding both handsides and Lie derivatives) the following covariance identity holds true

$$k\int_0^1 f(\pounds_\Xi \alpha, \Omega_s^{k-1})\,\mathrm{d}s + k(k-1)\int_0^1 f(\alpha, \pounds_\Xi \Omega_s, \Omega_s^{k-2})\,\mathrm{d}s = \mathrm{d}(\xi \rfloor \overset{(k)}{L}_{A\bar{A}}) \tag{5.1}$$

where we set Ξ for the generator of infinitesimal gauge transformation which projects over the spacetime vector field ξ and $\rfloor$ denotes the contraction of a vector field with an m-form.

Notice that we can use the connection ω_s ($\bar{A}$, respectively) to define the vertical part of Ξ

$$(\xi^s_{(V)})^A = \Xi^A + (\omega_s)^A_\mu \xi^\mu \qquad [(\bar{\xi}_{(V)})^A = \Xi^A + \bar{A}^A_\mu \xi^\mu, \text{ respectively}] \tag{5.2}$$

The vertical part of a vector field transforms with the adjoint (Ad) representation, so that we can define a section in $\Lambda^0(P;\mathfrak{g})$ by setting

$$\xi^s_{(V)} = (\xi^s_{(V)})^A \, T_A \otimes \mathbb{I} \tag{5.3}$$

Let us also introduce the following notation for the contraction of the curvature along a spacetime vector field, which turns out to be a section of $\Lambda^1(P;\mathfrak{g})$

$$\xi \rfloor \Omega_s = \xi^\lambda (\Omega_s)^A_{\lambda\mu} \, T_A \otimes \mathrm{d}x^\mu \tag{5.4}$$

and the 0-form

$$\xi \rfloor \alpha = \xi^\lambda \alpha^A_\lambda \, T_A \otimes \mathbb{I} \tag{5.5}$$

Using this notation we can express the Lie derivative of the connection as

$$\pounds_\Xi \omega_s = \xi \rfloor \Omega_s + \nabla^s \xi^s_{(V)} \tag{5.6}$$

The Nöther current is easily obtained from (5.1) as

$$\begin{aligned}
\mathcal{E}(L^{(k)}_{A\bar{A}}, \Xi) =& k(k-1) \int_0^1 f(\pounds_\Xi \omega_s, \alpha, \Omega_s^{k-2}) \, \mathrm{d}s - k \int_0^1 \xi \rfloor f(\alpha, \Omega_s^{k-1}) \, \mathrm{d}s = \\
=& k(k-1) \int_0^1 f(\xi \rfloor \Omega_s + \nabla^s \xi^s_{(V)}, \alpha, \Omega_s^{k-2}) \, \mathrm{d}s - k \int_0^1 f(\xi \rfloor \alpha, \Omega_s^{k-1}) \, \mathrm{d}s + \\
&+ k(k-1) \int_0^1 f(\alpha, \xi \rfloor \Omega_s, \Omega_s^{k-2}) \, \mathrm{d}s = \\
=& k(k-1) \int_0^1 f(\nabla^s \xi^s_{(V)}, \alpha, \Omega_s^{k-2}) \, \mathrm{d}s - k \int_0^1 f(\xi \rfloor \alpha, \Omega_s^{k-1}) \, \mathrm{d}s
\end{aligned} \tag{5.7}$$

This Nöther current can be covariantly integrated by parts

$$\mathcal{E}(L^{(k)}_{A\bar{A}}, \Xi) = \tilde{\mathcal{E}}(L^{(k)}_{A\bar{A}}, \Xi) + \mathrm{d}\mathcal{U}(L^{(k)}_{A\bar{A}}, \Xi) \tag{5.8}$$

to define the *superpotential*

$$\mathcal{U}(L^{(k)}_{A\bar{A}}, \Xi) = k(k-1) \int_0^1 f(\xi^s_{(V)}, \alpha, \Omega_s^{k-2}) \, \mathrm{d}s \tag{5.9}$$

and the *reduced current*

$$\tilde{\mathcal{E}}(L^{(k)}_{A\bar{A}},\Xi) = -k(k-1)\int_0^1 f(\xi^s_{(V)},\nabla^s\alpha,\Omega_s^{k-2})\,\mathrm{d}s - k\int_0^1 f(\xi\rfloor\alpha,\Omega_s^{k-1})\,\mathrm{d}s \tag{5.10}$$

One can directly check that the reduced current vanishes on-shell by simply expanding $\Omega_s = sF + (1-s)\bar{F} - s(1-s)(\alpha^2)$, $\xi^s_{(V)} = \bar{\xi}_{(V)} + s\xi\rfloor\alpha$, $\nabla^s\alpha = \bar{\nabla}\alpha + s[\alpha,\alpha] \equiv \frac{\mathrm{d}\Omega_s}{\mathrm{d}s}$ and then by integrating by parts with respect to the derivative $\frac{\mathrm{d}\Omega_s}{\mathrm{d}s}$. The result factorizes through field equations.

The superpotential (5.9) can be specialized to the case $k=2$ to obtain

$$\mathcal{U}(L^{(2)}_{A\bar{A}},\Xi) = f(\xi_{(V)} + \bar{\xi}_{(V)},\alpha) \tag{5.11}$$

as already computed in [**19**].

We remark that the non-covariant Lagrangians are not invariant with respect to gauge transformations. Only the total Lagrangian $L^{(k)}_{A\bar{A}} = L^{(k)}_{A} - L^{(k)}_{\bar{A}} + \mathrm{d}\Delta^{(k)}$ is covariant. Nöther theorem and superpotential theory apply to the covariant Lagrangian only. In other words, despite the superpotential $\mathcal{U}(L^{(k)}_{A\bar{A}},\Xi)$ receives contributions from $L^{(k)}_{A}$, $L^{(k)}_{\bar{A}}$ and $\Delta^{(k)}$, the corresponding splitting of the superpotential $\mathcal{U}(L^{(k)}_{A\bar{A}},\Xi) = \mathcal{U}(L^{(k)}_{A},\Xi) - \mathcal{U}(L^{(k)}_{\bar{A}},\Xi) + \mathcal{U}(\mathrm{d}\Delta^{(k)},\Xi)$ is not endowed with a fundamental meaning, each term depending on the trivialization (or equivalently on the gauge fixing).

For example, in the case $k=2$ the contributions are

$$\begin{cases} \mathcal{U}(L^{(2)}_{A},\Xi) = f(A,\xi_{(V)}) \\ \mathcal{U}(L^{(2)}_{\bar{A}},\Xi) = f(\bar{A},\bar{\xi}_{(V)}) \\ \mathcal{U}(\mathrm{d}\Delta^{(2)},\Xi) = f(A,\bar{\xi}_{(V)}) - f(\bar{A},\xi_{(V)}) \end{cases} \tag{5.12}$$

none of which is gauge invariant whilst their sum is.

6. BF Theory

As suggested in [**21**] we can change variables in the space of fields so to write Chern-Simons theory in a form which can be later specialized as a BF theory (see [**22**], [**23**], [**24**], [**25**], [**26**]), i.e. a gauge natural theory for a connection and a section of $\Lambda^1(P;\mathfrak{g})$, see [**6**].

The 3-dimensional case has been already dealt with in [**21**]. Here we are able to generalize our previous results to the case of any odd dimension. Let us first define an *average connection*

$$\tilde{\omega} := \omega_\sigma = \sigma A + (1-\sigma)\bar{A} \tag{6.1}$$

with $\sigma \in [0, 1]$ and the *relative field* $\alpha = A - \bar{A}$. We shall use these two fields $(\tilde{\omega}, \alpha)$ as fundamental fields in place of the original fields A and $\bar{A}$. This provides a one-parameter family of mutally equivalent topological field theories, the so-called BF type theories.

The converse transformation is

$$\begin{cases} A = \tilde{\omega} + (1 - \sigma)\alpha \\ \bar{A} = \tilde{\omega} - \sigma\alpha \end{cases} \tag{6.2}$$

The interpolating connection ω_s can be also expressed in terms of $\tilde{\omega}$ and α as

$$\omega_s = \tilde{\omega} + (s - \sigma)\alpha \tag{6.3}$$

so that we obtain

$$\Omega_s = \tilde{\Omega} + (s - \sigma)\tilde{\nabla}\alpha + (s - \sigma)^2(\alpha^2) \tag{6.4}$$

where $\tilde{\Omega}$ is the curvature of $\tilde{\omega}$. Hence we can express the Lagrangian as

$$L^{(k)}_{A\bar{A}} = k \int_0^1 f(\alpha, \Omega_s^{k-1})\, \mathrm{d}s \tag{6.5}$$

where Ω_s is expressed in terms of the fundamental fields and their derivatives as in (6.4).

For $k = 2$ we obtain the results of **[21]**, i.e.

$$L^{(2)}_{A\bar{A}} = f(\alpha, 2\tilde{\Omega} + (1 - 2\sigma)\tilde{\nabla}\alpha + \tfrac{1}{3}(1 + 3\sigma^2 - 3\sigma)[\alpha, \alpha]) \tag{6.6}$$

We stress that here σ has to be interpreted as a parameter to be specialized *a priori*.

Field equations can be easily obtained by

$$\begin{aligned} \mathbb{E}(L^{(k)}_{A\bar{A}}) =& kf(\delta A, F^{k-1}) - kf(\delta\bar{A}, \bar{F}^{k-1}) = \\ =& kf(\delta\tilde{\omega}, F^{k-1} - \bar{F}^{k-1}) + kf(\delta\alpha, (1 - \sigma)F^{k-1} + \sigma\bar{F}^{k-1}) \end{aligned} \tag{6.7}$$

where F and $\bar{F}$ are meant to be expressed by

$$\begin{cases} F = \tilde{\Omega} + (1 - \sigma)\tilde{\nabla}\alpha + (1 - \sigma)^2(\alpha^2) \\ \bar{F} = \tilde{\Omega} - \sigma\tilde{\nabla}\alpha + \sigma^2(\alpha^2) \end{cases} \tag{6.8}$$

Again for $k = 2$ we obtain the known results

$$\mathbb{E}(L^{(2)}_{A\bar{A}}) = 2f(\delta\tilde{\omega}, \tilde{\nabla}\alpha - (2\sigma - 1)(\alpha^2)) + 2f(\delta\alpha, \tilde{\Omega} + \sigma(1 - \sigma)(\alpha^2) + [\tilde{\nabla}\alpha - (2\sigma - 1)(\alpha^2)]) \tag{6.9}$$

The superpotential can be recast as

$$\mathcal{U}(L^{(k)}_{A\bar{A}}, \Xi) = k(k-1)\int_0^1 f(\xi^s_{(V)}, \alpha, \Omega^{k-2}_s)\, \mathrm{d}s \tag{6.10}$$

which for $k = 2$ specializes to

$$\mathcal{U}(L^{(2)}_{A\bar{A}}, \Xi) = f(\tilde{\xi}_{(V)} + (1-2\sigma)\xi \rfloor \alpha, \alpha) \tag{6.11}$$

Here $\tilde{\xi}_{(V)}$ denotes the vertical part of the symmetry generator Ξ with respect to the connection $\tilde{\omega}$.

7. Conclusions and Perspectives

We have here extended the formulation of covariant Chern-Simons theories to an arbitrary odd dimension and investigated their relation with BF theories. The introduction of a vacuum reference field $\bar{A}$ allows to define a global gauge covariant Lagrangian depending on two dynamical fields A and $\bar{A}$ both obeing the (global) field equations defining Chern-Simons models. We stress that the reference field is not to be interpreted as a background (in the sense in which Minkowski spacetime is a background in particle field theory) and it can be given a suitable physical interpretation (see [**14**]).

The resulting global Lagrangian for Chern-Simons theory is quite similar to the covariant first order Lagrangian for GR (see [**14**], [**15**], [**16**]) despite the fact that it is obtained by relying on quite different motivations (see [**17**]).

The understanding of the geometric relation between conservation laws for gauge-like Chern-Simons theories and Chern-Simons gravity is, at least to some extent, still partially obscure. In fact, these two models are gauge natural field theories for two different structure groups. We begin to understand this relation in some example (see, e.g., [**8**], [**19**]), but there is yet no general framework able to deal with these sort of transformations in full generality. This will form the subject of future investigations.

Appendix A. Lagrangian Splittings Induced by Homotopies in the Space of Connections

We shall review the results presented in [**18**] here suitably adapted to our case,i.e. a field theory on the bundle of connections.

Let us first introduce some notation. Let A and $\bar{A}$ be two connections on a principal bundle P and let us set $\alpha = A - \bar{A}$, which is a section of the model vector bundle $\Lambda^1(P;\mathfrak{g})$. We define

$$\omega_s := \bar{A} + s\alpha = sA + (1-s)\bar{A} \tag{A.1}$$

for the *interpolating connection*, s being any real number.

Let $\Omega_s := \mathrm{d}\omega_s + \frac{1}{2}[\omega_s, \omega_s]$ denote the curvature of the interpolating connection ω_s. We stress that Ω_s differs from the convex interpolation of the curvatures $(F, \bar{F})$ of the connections $(A, \bar{A})$, respectively. In fact, one has

$$\begin{aligned} \Omega_s =& \bar{F} + s\bar{\nabla}\alpha + s^2(\alpha^2) = \\ =& sF + (1-s)\bar{F} + s(s-1)(\alpha^2) \end{aligned} \tag{A.2}$$

where $\bar{\nabla}\alpha := \mathrm{d}\alpha + [\bar{A}, \alpha]$ denotes the *exterior gauge-covariant differential* of α with respect to the connection $\bar{A}$.

Let us consider a first order (not necessarily gauge covariant or global) Lagrangian $L = L(A, F)$. Its first variation formula reads as

$$\begin{aligned} \delta L =& \frac{\partial L}{\partial A}\delta A + \frac{\partial L}{\partial F}\delta F = \frac{\partial L}{\partial A}\delta A + \frac{\partial L}{\partial F}\nabla\delta A = \\ =& < \mathbb{E}(L) \mid \delta A > + \mathrm{d} < \mathbb{F}(L) \mid \delta A > \end{aligned} \tag{A.3}$$

where $\nabla\delta A$ denotes the exterior gauge-covariant differential with respect to the connection A and we set

$$\begin{cases} < \mathbb{E}(L) \mid \delta A >= \left(\dfrac{\partial L}{\partial A} - \nabla\dfrac{\partial L}{\partial F}\right)\delta A \\ < \mathbb{F}(L) \mid \delta A >= \dfrac{\partial L}{\partial F^A_{\mu\nu}}\delta A^A_\mu \, \mathrm{d}\sigma_\nu \end{cases} \tag{A.4}$$

where $\mathrm{d}\sigma_\nu$ is the canonical local basis for $(m-1)$-forms on M induced by coordinates.

Let us now consider the auxiliary Lagrangian

$$\tilde{\Lambda} := \int_0^1 \frac{\mathrm{d}L_s}{\mathrm{d}s}\,\mathrm{d}s \tag{A.5}$$

where we set $L_s = L(\omega_s, \Omega_s)$. This auxiliary Lagrangian can be computed explicitely as

$$\tilde{\Lambda} = L_1 - L_0 \tag{A.6}$$

by direct integration of $(A.5)$ and by using the first variation formula $(A.3)$

$$\tilde{\Lambda} = \int_0^1 \left(\frac{\partial L}{\partial \omega_s} \frac{\mathrm{d}\omega_s}{\mathrm{d}s} + \frac{\partial L}{\partial \Omega_s} \frac{\mathrm{d}\Omega_s}{\mathrm{d}s} \right) \mathrm{d}s = \int_0^1 \left(\frac{\partial L}{\partial \omega_s} \alpha + \frac{\partial L}{\partial \Omega_s} \nabla^s \alpha \right) \mathrm{d}s \tag{A.7}$$

where we set $\nabla^s \alpha := \bar{\nabla}\alpha + s[\alpha, \alpha]$ for the exterior gauge-covariant differential with respect to the interpolating connection ω_s. By a gauge-covariant integration by part and comparing with first the variation formula $(A.3)$ we obtain

$$\tilde{\Lambda} = \int_0^1 < \mathbb{E}(L_s) \mid \alpha > \ \mathrm{d}s + \mathrm{d} \int_0^1 < \mathbb{F}(L_s) \mid \alpha > \ \mathrm{d}s \tag{A.8}$$

By finally defining

$$\begin{cases} \Lambda = \int_0^1 < \mathbb{E}(L_s) \mid \alpha > \ \mathrm{d}s \\ \Delta = - \int_0^1 < \mathbb{F}(L_s) \mid \alpha > \ \mathrm{d}s \end{cases} \tag{A.9}$$

one easily obtains the Lagrangian splitting

$$\Lambda = L_1 - L_0 + \mathrm{d}\Delta \tag{A.10}$$

Let us remark the following: (i) $L_1 = L(A, F)$ and $L_0 = L(\bar{A}, \bar{F})$; (ii) Λ has the same field equations of L (one for A and one for $\bar{A}$); (iii) Λ is obtained by integrating the field equations of the original Lagrangian L along the interpolating homotopy.

This method of producing an equivalent Lagrangian which splits is particularly effective for Chern-Simons theory. In fact, the method does not rely on any assumption of gauge covariance on the original Lagrangian (so that it can be applied also to the Chern-Simons non-covariant Lagrangians). The resulting Lagrangian Λ relies on field equations (which for Chern-Simons theory *are* in fact gauge covariant) and it is global and gauge covariant by construction.

As we shall prove in the Appendix B, field equations for the non-covariant Chern-Simons Lagrangians are in the form

$$< \mathbb{E}(L_A^{(k)}) \mid \delta A >= k f(\delta A, F^{k-1}) \tag{A.11}$$

which produces the Lagrangian

$$\Lambda = k \int_0^1 f(\alpha, \Omega_s^{k-1})\, \mathrm{d}s \tag{A.12}$$

which, in turn, coincides with the covariant Chern-Simons Lagrangian $L^{(k)}_{A\bar{A}}$ given by (4.1).

The ensuing splitting (A.10) reads now as

$$L^{(k)}_{A\bar{A}} = L^{(k)}_{A} - L^{(k)}_{\bar{A}} + \mathrm{d}\Delta^{(k)} \tag{A.13}$$

with the form

$$\Delta^{(k)} = -\sum_{i=0}^{k-2} \frac{(-1)^i k!(k-1)!}{(k-i-2)!(k+i)!} \int_0^1 f(\alpha, \omega_s, (\omega_s^2)^i, (\Omega_s)^{k-i-2})\, \mathrm{d}s \tag{A.14}$$

where we used the explicit formula for $\mathbb{F}(L^{(k)}_{A})$ computed in Appendix B below.

The main reason which justifies the use of the results of **[18]** is that they allow to explicitely compute the divergence term Δ in the splitting (A.13), which is in fact rather tedious to be computed directly in the generic dimension case; see **[17]**.

Appendix B. Field Equations for the Covariant Lagrangian

We shall here compute field equations for the covariant Chern-Simons Lagrangian $L^{(k)}_{A\bar{A}}$ introduced in (4.1). Of course one could simply notice that because of the Lagrangian decomposition (A.10) field equations of $L^{(k)}_{A\bar{A}}$ have to coincide with field equations of the non-covariant Lagrangians $L^{(k)}_{A}$, which would simplify the computation considerably. However, we shall here compute field equations for the covariant Lagrangian directly, in order to become more familiar with the tricks that are used in this paper for conservation laws and also in order to definitely clarify the relation between the trasgression technique we used when defining the covariant Lagrangian $L^{(k)}_{A\bar{A}}$ and the homotopic technique used in **[18]** in order to construct the Lagrangian splitting (A.10); one needs in fact to know that the covariant Lagrangian $L^{(k)}_{A\bar{A}}$ is precisely in the form (4.1). Moreover, we compute also the Poincaré-Cartan morphism which will be used to express Nöther currents and conservation laws.

We start by considering the Lagrangian variation

$$
\begin{aligned}
\delta L^{(k)}_{A\bar{A}} =& k \int_0^1 f(\delta\alpha, \Omega_s^{k-1})\,\mathrm{d}s + k(k-1)\int_0^1 f(\alpha, \delta\Omega_s, \Omega_s^{k-2})\,\mathrm{d}s = \\
=& k \int_0^1 \left[f(\delta\alpha, \Omega_s^{k-1}) + (k-1) f(\alpha, \nabla^s \delta\omega_s, \Omega_s^{k-2}) \right]\,\mathrm{d}s = \\
=& k \int_0^1 \left[f(\delta\alpha, \Omega_s^{k-1}) + (k-1) f(\nabla^s \alpha, \delta\omega_s, \Omega_s^{k-2}) \right]\,\mathrm{d}s + \\
& - k(k-1)\int_0^1 \mathrm{d}f(\alpha, \delta\omega_s, \Omega_s^{k-2})\,\mathrm{d}s = \mathbb{E}(L^{(k)}_{A\bar{A}}) + \mathrm{d}\mathbb{F}(L^{(k)}_{A\bar{A}})
\end{aligned}
\tag{B.1}
$$

where Bianchi identities $\nabla^s\Omega_s = 0$ were used and we have set

$$
\begin{cases}
\mathbb{E}(L^{(k)}_{A\bar{A}}) = k \int_0^1 \left[f(\delta\alpha, \Omega_s^{k-1}) + (k-1) f(\nabla^s\alpha, \delta\omega_s, \Omega_s^{k-2}) \right]\,\mathrm{d}s \\
\mathbb{F}(L^{(k)}_{A\bar{A}}) = k(k-1)\int_0^1 f(\delta\omega_s, \alpha, \Omega_s^{k-2})\,\mathrm{d}s
\end{cases}
\tag{B.2}
$$

Let us now expand the Euler-Lagrange morphism $\mathbb{E}(L^{(k)}_{A\bar{A}})$ as follows:

$$
\begin{aligned}
\mathbb{E}(L^{(k)}_{A\bar{A}}) =& k \int_0^1 \left[f(\delta\alpha, \Omega_s^{k-1}) + (k-1) f(\delta\bar{A}, \nabla^s\alpha, \Omega_s^{k-2}) + \right. \\
& \left. + (k-1) f(\delta\alpha, \Omega_s - \bar{F} + s^2(\alpha^2), \Omega_s^{k-2}) \right]\,\mathrm{d}s = \\
=& k^2 \int_0^1 f(\delta A, \Omega_s - a\bar{F} + a s^2(\alpha^2), \Omega_s^{k-2})\,\mathrm{d}s + \\
& - k^2 \int_0^1 f(\delta\bar{A}, \Omega_s - a\nabla^s\alpha - a\bar{F} + a s^2(\alpha^2), \Omega_s^{k-2})\,\mathrm{d}s
\end{aligned}
\tag{B.3}
$$

where $a = \frac{k-1}{k}$.

Theorem (B.4): *The following holds:*

$$
k^2 \int_0^1 f(\delta A, \Omega_s - a\bar{F} + a s^2(\alpha^2), \Omega_s^{k-2})\, ds = k f(\delta A, F^{k-1})
\tag{B.5}
$$

Proof: *first expand $\Omega_s = sF + (1-s)\bar{F} + s(s-1)(\alpha^2)$, then use the power formula and recall that*

$$
\int_0^1 s^m (1-s)^n\, ds = \frac{m!n!}{(m+n+1)!}
\tag{B.6}
$$

as one can prove by a multiple integration by parts. The left hand side of equation $(B.5)$ *can thus be recast as:*

$$\begin{aligned}
k^2\Big[&\sum_{i=0}^{k-2}\sum_{j=0}^{k-i-2}(-1)^i\binom{k-2}{i}\binom{k-i-2}{j}\frac{(k-j-1)!(i+j)!}{(k+i)!}+\\
&+\frac{1}{k}\sum_{i=0}^{k-2}\sum_{j=1}^{k-i-1}(-1)^i\binom{k-2}{i}\binom{k-i-2}{j-1}\frac{(k-j-1)!(i+j-1)!}{(k+i-1)!}+\\
&-\sum_{i=0}^{k-2}\sum_{j=1}^{k-i-1}(-1)^i\binom{k-2}{i}\binom{k-i-2}{j-1}\frac{(k-j)!(i+j-1)!}{(k+i)!}+\\
&-\frac{2k-1}{k}\sum_{i=1}^{k-1}\sum_{j=0}^{k-i-1}(-1)^i\binom{k-2}{i-1}\binom{k-i-1}{j}\frac{(k-j)!(i+j-1)!}{(k+i)!}+\\
&+\sum_{i=1}^{k-1}\sum_{j=0}^{k-i-1}(-1)^i\binom{k-2}{i-1}\binom{k-i-1}{j}\frac{(k-j-1)!(i+j-1)!}{(k+i-1)!}\Big]\cdot\\
&\quad\cdot f(\delta\alpha, F^{k-i-j-1}, \bar{F}^j, (A^2)^i)
\end{aligned} \tag{B.7}$$

The first term is the only one which contributes to the term with $i = 0$ *and* $j = 0$ *and it produces exactly* $kf(\delta A, F^{k-1})$ *while all the other contributions cancel away.* ∎

Corollary (B.8): *one has*

$$k^2\int_0^1 f(\delta\bar{A}, \Omega_s - a\nabla^s\alpha - a\bar{F} + as^2(\alpha^2), \Omega_s^{k-2}) = kf(\delta\bar{A}, \bar{F}^{k-1}) \tag{B.9}$$

Proof: *just change the integration variable* $\sigma = 1 - s$ *and use theorem* $(B.4)$. ∎

Hence we have obtained

$$\mathbb{E}(L^{(k)}_{A\bar{A}}) = kf(\delta A, F^{k-1}) - kf(\delta\bar{A}, \bar{F}^{k-1}) \tag{B.10}$$

As far as the Poincaré-Cartan morphism $\mathbb{F}(L^{(k)}_{A\bar{A}})$ is concerned we expand $\Omega_s = sF + (1-$

$s)\bar{F} + s(s-1)(\alpha^2)$ as above and obtain

$$\begin{aligned}\mathbb{F}(L^{(k)}_{A\bar{A}}) =& k(k-1)\int_0^1 f(\delta\omega_s, \alpha, \Omega_s^{k-2})\, \mathrm{d}s = \\ =& k(k-1)\sum_{i=0}^{k-2}\sum_{j=0}^{k-i-2}(-1)^i \binom{k-2}{i}\binom{k-i-2}{j} \cdot \\ &\cdot \left[\frac{(k-j-1)!(i+j)!}{(k+i)!} f(\delta A, \alpha, (\alpha^2)^i, \bar{F}^j, F^{k-i-j-2} + \right. \\ &\left. + \frac{(k-j-2)!(i+j+1)!}{(k+i)!} f(\delta\bar{A}, \alpha, (\alpha^2)^i, \bar{F}^j, F^{k-i-j-2})\right]\end{aligned} \tag{B.11}$$

Then specializing $\bar{A} = 0$ in $(B.11)$ we obtain the Poincaré-Cartan morphism of the local Lagrangians:

$$\mathbb{F}(L^{(k)}_A) = \sum_{i=0}^{k-2}(-1)^i \frac{k!(k-1)!}{(k-i-2)!(k+i)!} f(\delta A, A, (A^2)^i, F^{k-i-2}) \tag{B.12}$$

which agrees with the prescription $(A.14)$.

Finally we can specialize everything to the specific cases $k = 2, 3$ (see **[17]**).

For $k = 2$ we have the covariant Lagrangian

$$L^{(2)}_{A\bar{A}} = 2\int_0^1 f(\alpha, \Omega_s)\, \mathrm{d}s = f(\alpha, F) + f(\alpha, \bar{F}) - \tfrac{1}{3} f(\alpha, \alpha^2) \tag{B.13}$$

the non-covariant Lagrangians

$$L_A = f(A, F) - \tfrac{1}{3} f(A, A^2) \tag{B.14}$$

the field equations (see $(B.10)$)

$$\mathbb{E}(L^{(2)}_{A\bar{A}}) = 2f(\delta A, F) - 2f(\delta\bar{A}, \bar{F}) = \mathbb{E}(L^{(2)}_A) - \mathbb{E}(L^{(2)}_{\bar{A}}) \tag{B.15}$$

the Poincaré-Cartan morphism (see $(B.11)$)

$$\mathbb{F}(L^{(2)}_{A\bar{A}}) = f(\delta A, \alpha) + f(\delta\bar{A}, \alpha) \quad \Rightarrow \mathbb{F}(L^{(2)}_A) = f(\delta A, A) \tag{B.16}$$

and finally the splitting $L^{(2)}_{A\bar{A}} = L^{(2)}_A - L^{(2)}_{\bar{A}} + d\Delta^{(2)}$ with (see $(A.14)$)

$$\Delta^{(2)} = -f(A, \bar{A}) \tag{B.17}$$

Analogously, for $k = 3$ we have the covariant Lagrangian

$$\begin{aligned} L^{(3)}_{A\bar{A}} =& 3\int_0^1 f(\alpha, \Omega_s, \Omega_s)\,\mathrm{d}s = f(\alpha, F^2) + f(\alpha, \bar{F}^2) + f(\alpha, F, \bar{F}) + \\ &+ \tfrac{1}{10} f(\alpha, \alpha^2) - \tfrac{1}{2} f(\alpha, F, \alpha^2) - \tfrac{1}{2} f(\alpha, \bar{F}, \alpha^2) \end{aligned} \tag{B.18}$$

the non-covariant Lagrangians

$$L^{(3)}_A = f(A, F^2) + \tfrac{1}{10} f(A, A^2, A^2) - \tfrac{1}{2} f(A, F, A^2) \tag{B.19}$$

the field equations (see $(B.10)$)

$$\mathbb{E}(L^{(3)}_{A\bar{A}}) = 3f(\delta A, F^2) - 3f(\delta\bar{A}, \bar{F}^2) = \mathbb{E}(L^{(3)}_A) - \mathbb{E}(L^{(3)}_{\bar{A}}) \tag{B.20}$$

the Poincaré-Cartan morphism (see $(B.11)$)

$$\begin{aligned} \mathbb{F}(L^{(3)}_{A\bar{A}}) =& f(\delta A, \alpha, 2F + \bar{F} - \tfrac{1}{2}\alpha^2) + f(\delta\bar{A}, \alpha, F + 2\bar{F} - \tfrac{1}{2}\alpha^2) \quad \Rightarrow \\ &\Rightarrow \quad \mathbb{F}(L^{(3)}_A) = f(\delta A, A, 2F - \tfrac{1}{4}[A, A]) \end{aligned} \tag{B.21}$$

and finally the splitting $L^{(3)}_{A\bar{A}} = L^{(3)}_A - L^{(3)}_{\bar{A}} + d\Delta^{(3)}$ with (see $(A.14)$)

$$\Delta^{(3)} = -f(A, \bar{A}, F + \bar{F}) + \tfrac{1}{4} f(A, \bar{A}, [\alpha, \alpha]) + \tfrac{1}{4} f(A, \bar{A}, [A, \bar{A}]) \tag{B.22}$$

One can check these results against **[17]**. Here they are computed by specializing general formulae which are valid in any odd dimension, while they were there directly computed in those specific low dimensional cases, by means of *ad hoc* prescriptions for calculation.

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